

# TWO-SCALE ANALYSIS OF A NON STANDARD TRANSMISSION PROBLEM IN TWO-COMPONENT MEDIA

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**ABSTRACT.** In this short paper, periodic homogenization of a steady heat flow in two-component media with highly adhesive contact is performed via the two-scale convergence technique. Our micro-model is based on mass conservation for the heat flow in each phase with interfacial contact of adhesive type between these constituents. It is shown that the macroscopic model is a single phase elliptic equation.

## 1. INTRODUCTION

Diffusion processes in multi-component media with non standard transmission conditions play an important role in many areas of mechanical engineering such as reservoir petroleum, biomechanics, geophysics,... In this short paper, we shall deal with the homogenization of a steady heat flow in media made of two interacting systems with an interfacial barrier leading to a jump of heat flux (see for instance [12, 13, 18, 19, 20, 22] and the references therein). Our micro-model is a two-phase elliptic system consisting of two equations as a result of two different components and based on mass conservation for the heat flow in each phase, combined with the Fick's law and non standard transmission interfacial conditions between these constituents. The macro-model is derived by means of the two-scale convergence method [2]. It is shown that the overall behavior of the heat diffusion process such media obeys to a single phase equation.

The paper is organized as follows. In section 2, we set our microscopic model. We also give an existence and uniqueness result. At the end of this section we state the main result of this paper: Theorem 2.2. Section 3 is devoted to the proof of this Theorem.

## 2. SETTING OF THE PROBLEM AND THE MAIN RESULT

We consider  $\Omega$  a bounded and smooth domain of  $\mathbb{R}^N$  ( $N \geq 2$ ) and  $Y = ]0, 1[^N$  the generic cell of periodicity. Let  $Y_1, Y_2 \subset Y$  be two open disjoint subsets of  $Y$  such that  $Y = Y_1 \cup Y_2 \cup \Sigma$  where  $\Sigma = \partial Y_1 \cap \partial Y_2$ , assumed to be a smooth submanifold. We denote  $\nu$  the unit normal of  $\Sigma$ , outward to  $Y_1$ . For  $i = 1, 2$ , let  $\chi_i$  denote the characteristic function of  $Y_i$ , extended by  $Y$ -periodicity to  $\mathbb{R}^N$ . For  $\varepsilon > 0$ , we set

$$\Omega_i^\varepsilon = \{x \in \Omega : \chi_i(\frac{x}{\varepsilon}) = 1\} \quad i = 1, 2$$

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and  $\Sigma^\varepsilon = \partial\Omega_1^\varepsilon \cap \partial\Omega_2^\varepsilon$ . To avoid some unnecessary technical computations, we assume that  $\overline{\Omega_2^\varepsilon} \subset \Omega$  so that  $\Sigma^\varepsilon = \partial\Omega_2^\varepsilon$  and  $\partial\Omega_1^\varepsilon = \partial\Omega \cup \Sigma^\varepsilon$ . Let  $Z_i = \cup_{k \in \mathbb{Z}^N} (Y_i + k)$ . As in [2], we also assume that  $Z_1$  is smooth and a connected open subset of  $\mathbb{R}^N$ . Let  $A_1$  (resp.  $A_2$ ) denote the permeability of the medium  $Z_1$  (resp.  $Z_2$ ). Let  $f_i$  be a measurable function representing the thermal source density in the material  $\Omega_i^\varepsilon$ . Finally, let  $\gamma$  be the *non-rescaled* conductivity of the thin layer  $\Sigma^\varepsilon$ . We shall assume the followings:

- H1) The conductivity tensors  $A_1$  and  $A_2$  are continuous on  $\mathbb{R}^N$ ,  $Y$ -periodic and satisfy the ellipticity condition:

$$A_1 \xi \cdot \xi \geq C |\xi|^2, \text{ (resp. } A_2 \xi \cdot \xi \geq C |\xi|^2) \quad \forall \xi \in \mathbb{R}^N$$

where, here and in the rest of the paper,  $C$  denotes any positive constant independent of  $\varepsilon$ ;

- H2) The source terms  $f_1$  and  $f_2$  lie in  $L^2(\Omega)$ ;

- H3) The conductivity  $\gamma$  is a continuous function on  $\mathbb{R}^N$ ,  $Y$ -periodic and bounded from below:

$$\gamma(y) \geq C > 0, \quad \forall y \in \mathbb{R}^N.$$

In the sequel, we shall denote for  $x \in \mathbb{R}^N$ ,

$$\chi_i^\varepsilon(x) = \chi_i\left(\frac{x}{\varepsilon}\right), \quad A_i^\varepsilon(x) = A_i\left(\frac{x}{\varepsilon}\right), \quad \text{and} \quad \gamma^\varepsilon(x) = \varepsilon \gamma\left(\frac{x}{\varepsilon}\right).$$

In this paper, we shall study the multiscale modelling of the following set of equations:

$$-\operatorname{div}(A_1^\varepsilon \nabla u_1^\varepsilon) = f_1 \text{ in } \Omega_1^\varepsilon, \quad (2.1a)$$

$$-\operatorname{div}(A_2^\varepsilon \nabla u_2^\varepsilon) = f_2 \text{ in } \Omega_2^\varepsilon, \quad (2.1b)$$

$$A_1^\varepsilon \nabla u_1^\varepsilon \cdot \nu^\varepsilon - A_2^\varepsilon \nabla u_2^\varepsilon \cdot \nu^\varepsilon = \gamma^\varepsilon \text{ on } \Sigma^\varepsilon, \quad (2.1c)$$

$$u_1^\varepsilon = u_2^\varepsilon \text{ on } \Sigma^\varepsilon, \quad (2.1d)$$

$$u^\varepsilon = 0 \text{ on } \partial\Omega \quad (2.1e)$$

where  $\nu^\varepsilon$  stands for the unit normal of  $\Sigma^\varepsilon$  outward to  $\Omega_1^\varepsilon$ . Here,  $\Omega_i^\varepsilon$  represents the region with conductivity  $A_i^\varepsilon$ . In this connection, the quantities  $u_1^\varepsilon$  and  $u_2^\varepsilon$  represent the temperature in  $\Omega_1^\varepsilon$  and  $\Omega_2^\varepsilon$ , respectively. The equations (2.1a) and (2.1b) express the steady heat diffusion of the temperature in  $\Omega_1^\varepsilon$  and  $\Omega_2^\varepsilon$  respectively, with Fick's law  $J_i^\varepsilon = A_i^\varepsilon \nabla u_i^\varepsilon$  where  $J_i^\varepsilon$  is the diffusion flux. Here we consider the problem that describes steady heat diffusion processes in which at the local scale there is a jump of diffusion fluxes with continuous temperature between the two constituents due to the fact that the interface  $\Sigma^\varepsilon$  is an active membrane. Similar phenomena are also observed in biological systems [9], chemical reactions [8], [10], [21], continuum mechanics. Finally, (2.1e) is the homogeneous Dirichlet condition on the exterior boundary of  $\Omega$ .

The analysis of transmission conditions such as (2.1c) arise for example in the study of composite structures glued together by thin adhesive layers which are thermally or mechanically very different from the components [18, 19, 20, 22]. In modern material technology such composites are widely used but the numerical treatment of the mathematical model by finite elements methods is still difficult, requires the introduction of highly inhomogeneous meshes, and often leads to poor accuracy and numerical instability (see, e.g., Babuska and Suri [5]). A convenient

way to overcome this problem is to replace the thin layers by zero thickness interfaces between the composite's components. Then one has to define on such interfaces suitable transmission conditions which incorporates the thermal and mechanical properties of the original layers. Such a procedure can be rigorously justified by an asymptotic method and leads to the introduction of boundary value problems with nonlinear transmission conditions such as those in (1) (see for example [9, 18] and the references therein).

Let us denote for convenience

$$A^\varepsilon = \chi_1^\varepsilon A_1^\varepsilon + \chi_2^\varepsilon A_2^\varepsilon \text{ and } f^\varepsilon = \chi_1^\varepsilon f_1^\varepsilon + \chi_2^\varepsilon f_2^\varepsilon$$

The weak formulation of (2.1a)-(2.1e) is as follows: find  $u^\varepsilon = \chi_1^\varepsilon u_1^\varepsilon + \chi_2^\varepsilon u_2^\varepsilon \in H_0^1(\Omega)$ , such that for all  $v \in H_0^1(\Omega)$ , we have

$$\int_{\Omega} A^\varepsilon \nabla u^\varepsilon \nabla v \, dx = \int_{\Omega} f^\varepsilon v \, dx + \int_{\Sigma^\varepsilon} \gamma^\varepsilon v \, ds^\varepsilon \quad (2.2)$$

where  $dx$  and  $ds^\varepsilon$  denote respectively the Lebesgue measure on  $\mathbb{R}^N$  and the Hausdorff measure on  $\Sigma^\varepsilon$ .

Now we give an existence and uniqueness result of (2.2)

**Theorem 2.1.** *Let assumptions H1)-H3) be fulfilled. Then, for any sufficiently small  $\varepsilon > 0$ , there exists a unique couple  $u^\varepsilon \in H_0^1(\Omega)$ , solution of the weak problem (2.2), such that*

$$\|u^\varepsilon\|_{H_0^1(\Omega)} \leq C. \quad (2.3)$$

*Proof.* It is a straightforward application of the Lax Milgram Theorem. The only point to show is the continuity of the form

$$v \longmapsto \int_{\Omega} f^\varepsilon v \, dx + \int_{\Sigma^\varepsilon} \gamma^\varepsilon v \, ds^\varepsilon$$

on  $H_0^1(\Omega)$ . Indeed, let  $v \in H_0^1(\Omega)$ . From the trace Theorem and by the rescaling technique (see for example [16])  $\square$

Now, we are ready to state the main result of the paper:

**Theorem 2.2.** *Let  $u^\varepsilon \in H_0^1(\Omega)$  be the unique solution of the weak system (2.2). Then, there exist a unique  $u \in H^1(\Omega)$  and a subsequence of  $(u^\varepsilon)$  still denoted  $(w^\varepsilon)$  such that  $w^\varepsilon$  converges weakly in  $H^1(\Omega)$  to  $u$  which is the solution of the homogenized model:*

$$\begin{cases} -\operatorname{div}(A^{\text{hom}} \nabla u) = F \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases} \quad (2.4)$$

where  $A^{\text{hom}}$  and  $F$  are given in (3.9) and (3.2) respectively.

The remainder of this paper is devoted to the proof of this Theorem. To prove this result we shall employ the two-scale convergence technique. Let

**Definition 2.1** (G. Allaire [2]). *We say that a sequence  $(v^\varepsilon)$  in  $L^2(\Omega)$  is two-scale convergent to  $v_0 \in L^2(\Omega \times Y)$  (we write  $v^\varepsilon \xrightarrow{2-s} v_0$ ) if, for any admissible test function  $\varphi \in L^2(\Omega; \mathcal{C}_\#(Y))$ ,*

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} v^\varepsilon(x) \varphi\left(x, \frac{x}{\varepsilon}\right) dx = \int_{\Omega \times Y} v_0(x, y) \varphi(x, y) \, dx dy$$

where  $\mathcal{C}_\#(Y)$  is the space of all continuous functions on  $\mathbb{R}^N$  which are  $Y$ -periodic.

Let  $L_{\#}^2(Y)$  be the space of all  $Y$ -periodic functions belonging to  $L_{\text{loc}}^2(\mathbb{R}^N)$  and  $H_{\#}^1(Y)$  denotes the space of those functions together with their derivatives belonging to  $L_{\#}^2(Y)$ . The following result concerns the two-scale convergence of uniformly bounded sequences in the Sobolev space  $H^1(\Omega)$ . See [2, 3].

**Theorem 2.3.** *Let  $(v^\varepsilon)$  be a uniformly bounded sequence in  $H^1(\Omega)$  (resp.  $H_0^1(\Omega)$ ). Then there exists  $v_0 \in H^1(\Omega)$  (resp.  $H_0^1(\Omega)$ ) and  $v^* \in L^2(\Omega; H_{\#}^1(Y)/\mathbb{R})$  such that, up to a subsequence,  $v^\varepsilon \xrightarrow{2-s} v_0$  and  $\nabla v^\varepsilon \xrightarrow{2-s} \nabla v_0 + \nabla_y v^*$ . Furthermore for every  $\varphi \in \mathcal{D}(\Omega; C_{\#}^\infty(Y))$  we have*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\Sigma^\varepsilon} v^\varepsilon \varphi^\varepsilon ds^\varepsilon = \int_{\Omega \times \Sigma} v_0 \varphi dx ds$$

where  $C_{\#}^\infty(Y) = C^\infty(\mathbb{R}^N) \cap C_{\#}(Y)$  and  $\varphi^\varepsilon(x) = \varphi(x, \frac{x}{\varepsilon})$ .

### 3. PROOF OF THEOREM 2.2

In this section, we shall determine the limiting problem (2.4). First, thanks to the a priori estimates (2.3) and Theorem 2.3, there exist a subsequence of  $(u^\varepsilon)$ , still denoted  $(u^\varepsilon)$  and a unique  $u \in H_0^1(\Omega)$ ,  $u^* \in L^2(\Omega; H_{\#}^1(Y)/\mathbb{R})$  such that  $u^\varepsilon \xrightarrow{2-s} u$  and  $\nabla u^\varepsilon \xrightarrow{2-s} \nabla u + \nabla_y u^*$ . Now, let  $\varphi \in \mathcal{D}(\Omega)$  and  $\varphi^* \in \mathcal{D}(\Omega; C_{\#}^\infty(Y))$ . Set  $\varphi^\varepsilon(x) = \varphi(x) + \varepsilon \varphi^*(x, \frac{x}{\varepsilon})$  and take  $v = \varphi^\varepsilon$  as a function test in (2.2), we obtain

$$\int_{\Omega} A^\varepsilon \nabla u^\varepsilon \left( \nabla \varphi + \nabla_y \varphi^* \left( x, \frac{x}{\varepsilon} \right) \right) dx = \int_{\Omega} f^\varepsilon \varphi dx + \int_{\Sigma^\varepsilon} \gamma^\varepsilon \varphi ds^\varepsilon + \varepsilon R^\varepsilon \quad (3.1)$$

where

$$R^\varepsilon = \int_{\Omega} A^\varepsilon \nabla u^\varepsilon \nabla_x \varphi^* \left( x, \frac{x}{\varepsilon} \right) dx + \int_{\Sigma^\varepsilon} \gamma^\varepsilon \varphi^* \left( x, \frac{x}{\varepsilon} \right) ds^\varepsilon.$$

According to the assumptions H1)-H3) the vectorial functions  $\chi_i({}^t A_i \nabla \varphi)$  and  $\chi_i({}^t A_i \nabla \varphi^*)$   $i = 1, 2$  are admissible in the sense that it can be used as test functions w.r.t. the notion of two scale convergence. It follows that the limit of the l.h.s. of (3.1) is

$$\int_{\Omega \times Y} A(\nabla u + \nabla_y u^*)(\nabla \varphi + \nabla_y \varphi^*) dx dy$$

and that

$$\lim_{\varepsilon \rightarrow 0} \left( \int_{\Omega} f^\varepsilon(x) \varphi(x) dx + \int_{\Sigma^\varepsilon} \gamma^\varepsilon \varphi(x) ds^\varepsilon \right) = \int_{\Omega} F(x) \varphi(x) dx,$$

where

$$A(y) = \chi_1(y) A_1(y) + \chi_2(y) A_2(y)$$

and

$$F(x) = |Y_1| f_1(x) + |Y_2| f_2(x) + \int_{\Sigma} \gamma(y) ds(y). \quad (3.2)$$

Moreover, using again (2.3), it is easy to check that  $R^\varepsilon = O(1)$ . Thus, by collecting all the above limits we get the two-scale variational formulation:

$$\int_{\Omega \times Y} A(\nabla u + \nabla_y u^*)(\nabla \varphi + \nabla_y \varphi^*) dx dy = \int_{\Omega} F \varphi dx. \quad (3.3)$$

By denseness argument, equation (3.3) still holds true for any  $(\varphi, \varphi^*) \in H_0^1(\Omega) \times L^2(\Omega; H_{\#}^1(Y)/\mathbb{R})$ . Now, integrating by parts in (3.3) yields the following two-scale homogenized system:

$$-\operatorname{div}_y (A(\nabla u + \nabla_y u^*)) = 0 \text{ a.e. in } \Omega \times Y, \quad (3.4)$$

$$-\operatorname{div} \left( \int_Y A(\nabla u + \nabla_y u^*) dy \right) = F \text{ a.e. in } \Omega, \quad (3.5)$$

$$u = 0 \text{ a.e. on } \partial\Omega, \quad (3.6)$$

$$y \mapsto u^* \text{ } Y\text{-periodic}, \quad (3.7)$$

Let us denote for  $1 \leq j \leq N$ ,  $\omega_j \in H_{\#}^1(Y_1)/\mathbb{R}$  the unique solution to the following cell problem:

$$\begin{cases} -\operatorname{div}_y (A(\nabla_y \omega_j + e_j)) = 0 \text{ a.e. in } Y_1, \\ A(\nabla_y \omega_j + e_j) \cdot \nu = 0 \text{ a.e. on } \Sigma, \\ y \mapsto \omega_j \text{ } Y\text{-periodic}, \end{cases},$$

where  $(e_j)$  is the canonical basis of  $\mathbb{R}^N$ . Observe that Equations (3.4), (3.6) and (3.7) lead to the following relation:

$$u^*(x, y) = \sum_{j=1}^N \frac{\partial u}{\partial x_j}(x) \omega_j(y) + \tilde{u}(x) \quad (3.8)$$

where  $\tilde{u}(x)$  is any additive function independent of  $y$ . In the sequel, we shall denote for convenience

$$A^{\text{hom}} = (a_{ij})_{1 \leq i, j \leq N}, \quad a_{ij} = \int_Y A(\nabla_y \omega_i + e_i) \cdot (\nabla_y \omega_j + e_j) dy, \quad (3.9)$$

Let us mention that, in view of H1),  $A^{\text{hom}}$  is symmetric and positive definite, see [7]. Inserting (3.8) into (3.5) together with (3.6) yields the elliptic equation:

$$-\operatorname{div} (A^{\text{hom}} \nabla u) = F \text{ in } \Omega, \quad (3.10)$$

$$u = 0 \text{ on } \partial\Omega. \quad (3.11)$$

The proof of Theorem 2.2 is then achieved.

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